

# Exact inhomogeneous cosmologies whose source is a radiation-matter mixture with consistent thermodynamics.

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We derive a new class of exact solutions of Einstein's equations providing a physically plausible hydrodynamical description of cosmological matter in the radiative era, between nucleosynthesis and decoupling. The solutions are characterized by the Lemaître-Tolman-Bondi metric with a viscous fluid source, subjected to the following conditions: (a) the equilibrium state variables satisfy the equation of state of a mixture of an ultra-relativistic and a non-relativistic ideal gases, where the internal energy of the latter has been neglected, (b) the particle numbers of the mixture components are independently conserved, (c) the viscous stress is consistent with the transport equation and entropy balance law of Extended Irreversible Thermodynamics, with the coefficient of shear viscosity provided by Kinetic Theory. The fulfilment of (a), (b) and (c) restricts initial conditions in terms of an initial value function,  $\Delta_i^{(s)}$ , which in the limit of small density contrasts becomes the average of spatial gradients of the fluctuations of photon entropy per baryon in the initial hypersurface. For  $\Delta_i^{(s)} \neq 0$  and choosing the phenomenological coefficients of the “radiative gas” model, we have an interactive photon-baryon mixture under local thermal equilibrium, with radiation dominance and temperatures characteristic of the radiative era ( $10^6 \text{K} > T > 10^3 \text{K}$ ). Constraints on the observed anisotropy of the microwave cosmic radiation and the condition that decoupling occurs at  $T = T_D \approx 4 \times 10^3 \text{ K}$  yield an estimated value:  $|\Delta_i^{(s)}| \approx 10^{-8}$  which can be associated with a bound on primordial entropy fluctuations. The Jeans mass at decoupling is of the same order of magnitude as that of baryon dominated perturbation models ( $\approx 10^{16} M_\odot$ ).

## I. INTRODUCTION

The radiative era of cosmic evolution comprises the period from the end of primeval nucleosynthesis to the decoupling of matter and radiation (see refs [1] to [11]). A gross description of cosmological matter sources in this period is given by an interactive mixture of ideal relativistic and non relativistic gases (“radiation” and “matter”) in local thermal equilibrium (LTE).

The standard approach to this type of matter source is either a FLRW spacetime with equilibrium Kinetic Theory distributions [5], [6], [7], gauge invariant perturbations on a FLRW background [4], [5], [6], [8], [9], [10], and various types of hydrodynamical models, [12], [13], [14] which, in general, fail to incorporate a physically plausible description of the interaction between matter and radiation. Even if we argue that the universe is “almost FLRW” or “almost in thermal equilibrium”, the small deviations from equilibrium are extremely important, [1], [2], [4], [5], [6], [8], [9], to account for most interesting phenomena of cosmic evolution: nucleosynthesis, structure formation, abundance of relic gases, etc. Models with perfect fluid sources, whether hydrodynamic, [12], [13], or based on Kinetic Theory, [15], necessarily assume a quasi-static adiabatic and reversible evolution and thus, fail to incorporate into the resulting picture even small deviations from equilibrium.

Dissipative sources have been incorporated numerically within a purely FLRW geometry [16] or following a perturbative approach [17]. However, the literature still lacks an alternative hydrodynamical treatment, based on inhomogeneous exact solutions of Einstein's equations with dissipative sources and fully complying with the thermodynamics of a radiative gas within a transient regime. Ideally, such exact models should include all dissipative agents (heat flux, bulk and shear viscosity) and should be consistent with the theoretical framework of Extended Irreversible Thermodynamics (see refs [17] to [29]), thus satisfying suitable transport equations complying with causality, with phenomenological coefficients given by Kinetic Theory for this type of source. Since this general treatment would be mathematically untractable, we aim at the best possible approach based on exact solutions of Einstein's equations. Therefore, we have made the following simplifying assumptions: (a) the matter source is a fluid with shear viscosity but without heat conduction nor bulk viscosity, (b) the equilibrium state variables satisfy the equation of state of a mixture of relativistic and non-relativistic ideal gases, where the internal energy and pressure of the latter have been neglected, (c) the particle numbers of each mixture components is independently conserved, (d) we exclude dark matter and/or exotic particles and assume instead a tight coupling between photons (radiation) and baryons and electrons (matter), hence there is a common temperature for the mixture (LTE), while the microscopical interaction models

are the various processes of radiative transfer, [2], [6], [8], [23], [25], [26], [27]: Thomson scattering, brehmstrahlung, free-free absorption, etc. Although this type of interactions involve mostly photons and electrons, the dynamics of the matter component is governed by the baryons since the latter provide most of the rest mass content of non-relativistic matter (without dark matter).

Restrictions (b) and (c) are easy to justify: since the ratio of photons to baryons is such a large number ( $\approx 10^9$ ), we can truly ignore the pressure of non-relativistic matter. Also, after nucleosynthesis, in the temperature range  $T < 10^6$ , matter creation and annihilation processes balance each other and effectively cease to be dynamically important [5], [6]. On the other hand, the lack of heat conduction (restriction (a)) is more difficult to justify. It can be associated with an adiabatic (zero heat flux) but still irreversible evolution (nonzero viscosity), and can be a reasonable approximation on specific conditions. For example, for a radiative gas at higher temperatures shear viscosity dominates over heat conduction, but the latter becomes significant as the mixture cools [2]. The lack of bulk viscosity is a better approximation: it is negligible for a radiative gas in the temperature range  $10^3 \text{ K} < T < 10^6 \text{ K}$  that we are interested, [2], [17], [21], [23], [25], becoming important for higher temperatures (the mid relativistic regime where  $k_B T \approx mc^2$  [17], [28], [29]). However, we accept that ignoring these dissipative fluxes weakens the scope and validity of the models, but we argue that this is compensated by the simplification of the field equations, leading to exact forms for the equilibrium state variables and shear viscosity that still satisfy (under the restrictions mentioned) thermodynamically consistent relations.

The models we present are based on the spherically symmetric Lemaître-Tolman-Bondi metrics, usually associated with dust sources [30], [31]. However, this metric is compatible with a comoving fluid source with zero heat flux but with anisotropic stresses, which we describe as shear viscosity. Obviously, the lack of heat flux and 4-acceleration necessarily implies a very special shear viscous tensor whose divergence exactly balances the nonzero spatial gradient of the equilibrium pressure. Considering this metric and this source, we impose on the equilibrium state variables the equation of state for a mixture of ideal gases (under the restriction (b)). The field equations can be solved up to a quadrature, without having to make any assumption on the form of the shear viscous pressure. The latter, as well as all equilibrium state variables can be determined from the solution of the quadrature, up to two initial value functions that can be identified with the initial energy densities of the matter and radiation components. We consider only the case that would be equivalent to spacelike sections of zero curvature. A generalization of this class of exact solutions to the more general Szekeres-Szafron metrics admitting no isometries has been published recently [32], while the study of a non-relativistic ideal gas is considered in [33].

Once the field equations have been integrated, we define a set of initial value functions that gauge the deviation from homogeneity of the average of initial density contrasts. The terms involving various gradients of metric functions can be given in terms of these gauges, so that in the limit when the latter vanish a FLRW spacetime can always be obtained as the homogeneous (and reversible) subcase. In section VII we derive the conditions that the models must satisfy in order to be consistent with the theoretical framework of Irreversible Extended Thermodynamics, in the case where shear viscosity is the only dissipative agent and the coefficient of shear viscosity is that given by Kinetic Theory for the radiative gas [2], [23], [25], [26], [27], [29]. This leads to an entropy balance law and a suitable transport equation for shear viscosity that is satisfied for a specific functional form of the relaxation time. Conditions are given so that the latter quantity behaves as a relaxation parameter for an interactive cosmological mixture of matter and radiation. These conditions of thermodynamical consistency are then explicitly tested on the models, leading to a set of restrictions on the initial conditions (the latter given in terms of the gauges of initial density contrasts). The most relevant result is that thermodynamical consistency constrains an initial value adimensional function,  $\Delta_i^{(s)}$ , which in the limit of small density contrasts is approximately the average gradient of the photon entropy per baryon along the initial hypersurface  $t = t_i$ . An analogy is provided with the theory of perturbations on a FLRW background, whereby  $\Delta_i^{(s)} = 0$  is formally analogous to the definition of initially adiabatic perturbations in the synchronous gauge [4], [6], [9], [10]. The constraints on the observed anisotropy of the microwave cosmic background, as well as the condition that decoupling occurs at  $T = T_D \approx 4 \times 10^3 \text{ K}$ , leads to the estimated value  $|\Delta_i^{(s)}| \approx 10^{-8}$ . Since initial conditions of the radiative era should be traced to previous periods of cosmic evolution, this constraint can be related to maximal bounds on entropy fluctuations in primordial perturbations. Finally, we compute the Jeans mass associated with the thermodynamically consistent models, leading to a value similar to that obtained for baryon dominated perturbation models:  $M_J \approx 10^{16} M_\odot$ .

## II. INTERACTING MIXTURE OF RADIATION AND NON-RELATIVISTIC MATTER

A radiation-matter mixture can be described by a mixture of two ideal gases: one an ultra-relativistic gas of massless particles, the other a non-relativistic ideal monatomic gas with  $m$  being the mass of the particles. This is characterized by the total matter energy,  $\rho$  and pressure  $p$

$$\rho = mc^2 n^{(m)} + \frac{3}{2} n^{(m)} k_B T^{(m)} + 3 n^{(r)} k_B T^{(r)}, \quad (1a)$$

$$p = n^{(m)} k_B T^{(m)} + n^{(r)} k_B T^{(r)}, \quad (1b)$$

where  $k_B$  is Boltzmann's constant and  $n, T$  are particle number densities and temperatures of the two components, distinguished by the superindices  $(m)$  (“matter”) and  $(r)$  (“radiation”). If there is local thermal equilibrium (LTE) between the components, the latter interact and evolve with the same temperature:  $T^{(r)} = T^{(m)} = T$ . If the components are decoupled, each gas evolves with a different temperature.

Assuming LTE, if  $n^{(m)} \ll n^{(r)}$ , but the ratio  $mc^2/k_B T$  is not negligible, then equations (1) can be approximated by

$$\rho \approx mc^2 n^{(m)} + 3 n^{(r)} k_B T, \quad (2a)$$

$$p \approx n^{(r)} k_B T, \quad (2b)$$

an equation of state describing a radiation dominated mixture in which the presence of non-relativistic matter is dynamically important. If we assume non-relativistic matter to be made up of baryons (with  $m$  being a protonic mass) and since the ratio of baryons to photons  $n^{(m)}/n^{(r)} \approx 10^{-9}$  is a small number, the equation of state (2) is a reasonable approximation in the temperature range  $10^3 \leq T \leq 10^6$  K, characteristic of the “radiative era” from the end of nucleosynthesis to the transition between radiation to matter dominance, including the recombination and decoupling eras. At such temperatures, it is also safe to assume, [1], [3], [4], [5], [6] that electrons and photons interact mostly through Thomson scattering but creation and annihilation processes (bremsstrahlung and free-free absorption) roughly compensated one another so that particle number densities of the components of the mixture satisfy independent conservation laws. Once the decoupling of the matter-radiation mixture takes place at about  $T \approx 4 \times 10^3$  K, the assumption of LTE is no longer valid and interaction between components ceases. Equation of state (1) can also be approximated by a form similar (2) with the internal energy of radiation taking approximately the Stefan-Boltzmann law:  $\rho^{(r)} = a_B T^4$ , where  $a_B$  denotes the radiation constant. However, out of thermal equilibrium the Stefan-Boltzmann law is incompatible with the ideal gas equation of state.

Having in mind the conditions justifying (2), we will describe a matter-radiation mixture evolving along adiabatic but irreversible processes by the fluid tensor

$$T^{ab} = \rho u^a u^b + p h^{ab} + \Pi^{ab}, \quad (3)$$

$$h^{ab} = c^{-2} u^a u^b + g^{ab}, \quad u_a \Pi^{ab} = 0, \quad \Pi^a{}_a = 0,$$

where:  $\rho, p$  satisfy (2),  $u^a$  is the 4-velocity shared by radiation and matter,  $\Pi^{ab}$  is the shear viscous pressure tensor (a symmetric traceless tensor) which arises because of the matter-radiation interaction, and particle number densities satisfy the conservation laws

$$(n^{(m)} u^a)_{;a} = 0, \quad (n^{(r)} u^a)_{;a} = 0. \quad (4)$$

As mentioned previously, bulk viscosity is negligible within the temperature range we are interested in [23], [2], [25], [26], [27], [28], while even if neglect of heat conduction can be justified for relativistic temperatures [2], it does weaken the scope of the models. However, this restriction is compensated by the obtention of exact solutions that are still thermodynamically consistent.

### III. THE LEMAITRE-TOLMAN-BONDI METRICS

Consider (3) as the source of the “Lemaître-Tolman-Bondi” (LTB) metric ansatz, usually associated with spherically symmetric Lemaître-Tolman-Bondi dust solutions [30], [31]

$$ds^2 = -c^2 dt^2 + \frac{Y'^2}{1-F} dr^2 + Y^2 (d\theta^2 + \sin^2(\theta) d\phi^2) , \quad (5)$$

where  $Y = Y(t, r)$ ,  $F = F(r)$ , and a prime denotes partial derivative with respect  $r$ . Just as in the LTB dust solutions, we assume the coordinates in (5) to be comoving and the 4-velocity of the fluid source to be  $u^a = c \delta^a_t$ , a geodesic vector field, since  $\dot{u}_a \equiv u_{a;b} u^b = 0$ . Other kinematic invariants associated with (5) are the scalar expansion  $\Theta \equiv u^a_{;a}$ , and the shear tensor  $\sigma_{ab} \equiv u_{(a;b)} - (\Theta/3)h_{ab}$ , given in the coordinates of (5) by

$$\Theta = \frac{\dot{Y}'}{Y'} + \frac{2\dot{Y}}{Y} , \quad (6a)$$

$$\sigma^a_b = \mathbf{diag}[0, -2\sigma, \sigma, \sigma] , \quad \sigma \equiv \frac{1}{3} \left( \frac{\dot{Y}}{Y} - \frac{\dot{Y}'}{Y'} \right) , \quad (6b)$$

while the most general form of  $\Pi^a_b$  for the metric (5) is given by

$$\Pi^a_b = \mathbf{diag}[0, -2P, P, P] , \quad (6c)$$

where  $\dot{Y} \equiv u^a Y_{,a} = Y_{,t}$  and  $P = P(t, r)$  is an arbitrary function. Notice that a comoving and non-accelerating 4-velocity does not imply  $p' = 0$ , as in the perfect fluid case ( $G^\theta_\theta - G^r_r = 0$ ). As revealed by the momentum balance law:  $h_{ca} T^{ab}_{;b} = 0$ , applied to the viscous fluid source (3), we have

$$h_a^b (p_{,b} + \Pi_{bc;d} h^{cd}) = 0 \quad \Rightarrow \quad (p - 2P)' + 6P \frac{Y'}{Y} = 0 \quad (7a)$$

showing how the divergence of the shear viscous tensor exactly balances the nonzero pressure gradient. The energy balance,  $u_a T^{ab}_{;b} = 0$ , is given by

$$\dot{\rho} + (\rho + p)\Theta + \sigma_{ab} \Pi^{ab} = 0 \quad \Rightarrow \quad \dot{p} + \frac{4}{3}\Theta p + 6\sigma P = 0 \quad (7b)$$

illustrating how the term  $\sigma_{ab} \Pi^{ab} = 6\sigma P$  can be understood as an interaction term responsible for local energy exchange between matter and radiation.

Integration of the conservation laws (4) for (5) yields

$$n^{(m)} = n_i^{(m)} \left( \frac{Y_i}{Y} \right)^3 \frac{Y'_i/Y_i}{Y'/Y}, \quad n^{(r)} = n_i^{(r)} \left( \frac{Y_i}{Y} \right)^3 \frac{Y'_i/Y_i}{Y'/Y}, \quad (8)$$

where  $n_i^{(m)}$ ,  $n_i^{(r)}$  depend only on  $r$  and are the particle number densities of non-relativistic matter and radiation, evaluated along a suitable initial hypersurface labeled by  $t = t_i$ . The subindex “ $i$ ” affixed to any quantity, as  $Y_i$ , will denote henceforth initial value functions (functions of  $t, r$  evaluated along  $t = t_i$ ). It is important to state that our initial conditions do not refer to present cosmic time (usually labeled as  $t = t_0$ ), and so we will not use the subindex “ $0$ ”.

The spherically symmetric LTB metrics (5) are contained within a larger class of more general metrics (the Szekeres-Szafron metrics [30], [31]), admitting in general no isometries. The integration of the field equations for (5), given a source (3) satisfying (2), is examined in the next section. For the case of more general Szekeres-Szafron metrics, see [32].

#### IV. INTEGRATION OF THE FIELD EQUATIONS.

Einstein's field equations for (5) and (3) are

$$\kappa\rho = -\frac{\left[Y\left(\dot{Y}^2 + Fc^2\right)\right]'}{Y^2Y'} = -G^t{}_t, \quad (9a)$$

$$\kappa p = -\frac{\left[Y\left(\dot{Y}^2 + Fc^2\right) + 2Y^2\ddot{Y}\right]'}{3Y^2Y'} = \frac{1}{3}(2G^\theta{}_\theta + G^r{}_r), \quad (9b)$$

$$\kappa P = \frac{Y}{6Y'}\left[\frac{Y\left(\dot{Y}^2 + Fc^2\right) + 2Y^2\ddot{Y}}{Y^3}\right]' = \frac{1}{3}(G^\theta{}_\theta - G^r{}_r), \quad (9c)$$

where  $\kappa \equiv 8\pi G/c^2$ . Imposing on (9a) and (9b) the equation of state (2), using (8) and integrating with respect to  $r$  yields the following constraint

$$2Y(\dot{Y}^2 + Fc^2) + Y^2\ddot{Y} - \kappa mc^2 \int n_i^{(m)} Y_i^2 Y_i' dr = \lambda(t), \quad (10)$$

where  $\lambda(t)$  is an arbitrary integration function. It is important to remark that (10) follows only from (9a) and (9b) without involving (9c), *ie* it was not necessary to make any assumption regarding the form of  $P$  in order to obtain (10). A second integration of the field equations necessarily requires setting  $\lambda(t) = 0$  in (10), leading to

$$\dot{Y}^2 = \frac{\kappa}{Y}\left[M + W\left(\frac{Y_i}{Y}\right)\right] - Fc^2, \quad (11)$$

where

$$M = \int \rho_i^{(m)} Y_i^2 Y_i' dr, \quad \rho_i^{(m)} \equiv mc^2 n_i^{(m)}, \quad (12a)$$

$$W = \int \rho_i^{(r)} Y_i^2 Y_i' dr, \quad \rho_i^{(r)} \equiv 3n_i^{(r)} k_B T_i, \quad (12b)$$

so that  $\rho_i^{(m)}$ ,  $\rho_i^{(r)}$  respectively define the initial densities of the non-relativistic and relativistic components of the mixture.

In the remaining of the paper we restrict ourselves to  $F = 0$ , similar to the choice of spacelike sections of zero curvature in FLRW geometry, leaving the case  $F \neq 0$  for a future analysis. An explicit integral of (11) in this case is given by

$$\frac{3}{2}\sqrt{\mu}(t - t_i) = \sqrt{y + \epsilon}(y - 2\epsilon) - \sqrt{1 + \epsilon}(1 - 2\epsilon), \quad (13a)$$

where

$$\mu \equiv \frac{\kappa M}{Y_i^3}, \quad \epsilon \equiv \frac{W}{M}, \quad y \equiv \frac{Y}{Y_i}, \quad (13b)$$

It is possible to invert (13a), thus obtaining  $y = y(t, r)$  as a complicated, but closed analytic form, where the  $r$  dependence is contained in the functions  $\mu$ ,  $\epsilon$  appearing in (13b). However it turns out to be more convenient to use (11) and (13) to simplify the field equations and radial gradients of  $y$  in order to express all state and geometric variables in terms of  $y$  and suitable initial value functions related to those of (12) and (13b).

## V. THE STATE VARIABLES.

From (8) and (11)-(13) it is possible to obtain the state variables  $n^{(m)}$ ,  $n^{(r)}$ ,  $T$ ,  $\rho^{(m)}$ ,  $\rho^{(r)}$ ,  $p$ ,  $P$ . However, before doing so it is useful to define the averaged initial densities

$$\langle \rho_i^{(m)} \rangle \equiv \frac{\int \rho_i^{(m)} d(Y_i^3)}{Y_i^3} = \frac{3M}{Y_i^3}, \quad \langle \rho_i^{(r)} \rangle \equiv \frac{\int \rho_i^{(r)} d(Y_0^3)}{Y_i^3} = \frac{3W}{Y_i^3}, \quad (14)$$

averaged over the volume  $Y_i^3$ . Since the solutions allow for an arbitrary re-scaling of the radial coordinate, without loss of generality we can select  $Y_i = rR_i$ , where  $R_i$  is a characteristic constant length scale. Therefore, the volume  $Y_i^3$ , evaluated from the symmetry center,  $r = 0$ , to an arbitrary fluid layer  $r$ , can be characterized invariantly as the volume of the orbits of the rotation group  $SO(3)$  in the hypersurface  $t = t_i$ . In the Newtonian limit, the distance  $Y_i$  becomes the radius of the circular keplerian orbit in the field of (5).

Together with the averaged initial densities, we shall define the quantities  $\Delta_i^{(m)}$ ,  $\Delta_i^{(r)}$  given by

$$\Delta_i^{(m)} = \frac{\int [\rho_i^{(m)}]' Y_0^3 dr}{3M} = \frac{\rho_i^{(m)}}{\langle \rho_i^{(m)} \rangle} - 1 \quad \Rightarrow \quad \rho_i^{(m)} = \langle \rho_i^{(m)} \rangle [1 + \Delta_i^{(m)}], \quad (15a)$$

$$\Delta_i^{(r)} = \frac{\int [\rho_i^{(r)}]' Y_i^3 dr}{3W} = \frac{\rho_i^{(r)}}{\langle \rho_i^{(r)} \rangle} - 1 \quad \Rightarrow \quad \rho_i^{(r)} = \langle \rho_i^{(r)} \rangle [1 + \Delta_i^{(r)}], \quad (15b)$$

whose interpretation as effective initial density contrasts is discussed in the following section. Using (14) and (15) we can re-write  $\mu, \epsilon$  in (13b) as

$$\mu = \frac{\kappa}{3} \langle \rho_i^{(m)} \rangle = \frac{\kappa \rho_i^{(m)}}{3(1 + \Delta_i^{(m)})}, \quad \epsilon = \frac{\langle \rho_i^{(r)} \rangle}{\langle \rho_i^{(m)} \rangle} = \frac{\rho_i^{(r)}}{\rho_i^{(m)}} \frac{1 + \Delta_i^{(m)}}{1 + \Delta_i^{(r)}}. \quad (16)$$

The state variables  $n^{(m)}$ ,  $n^{(r)}$ ,  $\rho$ ,  $p$ ,  $P$  now follow by inserting (11) into (8) and (9), while  $T$  is obtained with the help of (2). This yields the following forms

$$n^{(m)} = \frac{n_i^{(m)}}{y^3 \Gamma}, \quad n^{(r)} = \frac{n_i^{(r)}}{y^3 \Gamma}, \quad (17a)$$

$$T = \frac{T_i}{y} \Psi, \quad (17b)$$

$$\rho = \rho^{(m)} + \rho^{(r)} = \left[ \frac{\rho_i^{(m)}}{y^3} + \frac{\rho_i^{(r)}}{y^4} \Psi \right] \frac{1}{\Gamma}, \quad (17c)$$

$$p = \frac{\rho_i^{(r)}}{3y^4} \frac{\Psi}{\Gamma}, \quad (17d)$$

$$P = \frac{\rho_i^{(r)}}{6y^4} \frac{\Phi}{\Gamma}, \quad (17e)$$

where the functions  $\Gamma$ ,  $\Psi$  and  $\Phi$  are given by

$$\Gamma \equiv \frac{Y'/Y}{Y'_i/Y_i} \quad \Psi \equiv 1 + \frac{(1-\Gamma)}{3(1+\Delta_i^{(r)})}, \quad \Phi \equiv 1 + \frac{(1-4\Gamma)}{3(1+\Delta_i^{(r)})}. \quad (18)$$

The solutions characterized by (2), (4), (5)-(18) become determinate once  $\Gamma$  above is obtained in terms of  $y$  from (13) for given initial value functions  $\rho_i^{(m)}, \rho_i^{(r)}$  (re-expressed in terms of the quantities  $\Delta_i^{(m)}, \Delta_i^{(r)}, \epsilon$ ). This transforms (18) into

$$\Gamma = 1 + 3A\Delta_i^{(m)} + 3B\Delta_i^{(r)}, \quad (19a)$$

$$\Psi = 1 - \frac{A\Delta_i^{(m)} + B\Delta_i^{(r)}}{1 + \Delta_i^{(r)}}, \quad (19b)$$

$$\Phi = \frac{-4A\Delta_i^{(m)} + (1-4B)\Delta_i^{(r)}}{1 + \Delta_i^{(r)}}, \quad (19c)$$

where

$$A = \frac{1}{3y^2} \left[ y^2 - 4\epsilon y - 8\epsilon^2 - \frac{\sqrt{y+\epsilon}}{\sqrt{1+\epsilon}} (1 - 4\epsilon - 8\epsilon^2) \right], \quad (20a)$$

$$B = \frac{\epsilon}{y^2} \left[ y + 2\epsilon - (1+2\epsilon) \frac{\sqrt{y+\epsilon}}{\sqrt{1+\epsilon}} \right], \quad (20b)$$

with  $\Delta_i^{(m)}, \Delta_i^{(r)}$  given by (15). The kinematic parameters  $\sigma, \Theta$  follow by inserting (11) with  $y = Y/Y_i$  and (19a) into (6a) and (6b)

$$\sigma = -\frac{\sqrt{\mu}\sqrt{y+\epsilon} \left[ A_{,y} \Delta_i^{(m)} + B_{,y} \Delta_i^{(r)} \right]}{y^2 \left[ 1 + 3 \left( A\Delta_i^{(m)} + B\Delta_i^{(r)} \right) \right]}, \quad (21)$$

$$\frac{\Theta}{3} = \frac{\sqrt{\mu}\sqrt{y+\epsilon} \left[ 1 + (3A + yA_{,y}) \Delta_i^{(m)} + (3B + yB_{,y}) \Delta_i^{(r)} \right]}{y^2 \left[ 1 + 3 \left( A\Delta_i^{(m)} + B\Delta_i^{(r)} \right) \right]}, \quad (22)$$

where  $A_{,y}, B_{,y}$  are the derivatives of  $A, B$  in (20) with respect to  $y$ . Given a set of initial conditions specified by  $\epsilon, \Delta_i^{(m)}, \Delta_i^{(r)}$ , equations (17) and (19)-(22) provide fully determined forms of the state and geometric variables as functions of  $y$  and the chosen initial conditions.

The solutions presented so far contain a FLRW particular case, obtained by setting in (11) and (12)  $n_i^{(m)} = \bar{n}_i^{(m)}, n_i^{(r)} = \bar{n}_i^{(r)}$  and  $T_i = \bar{T}_i$ , where  $\bar{n}_i^{(m)}, \bar{n}_i^{(r)}, \bar{T}_i$  are arbitrary positive constants. Under this parameter specialization, (13) holds with  $Y = R(t)f(r)$  (so that  $y = R(t)$ ) and (5) becomes a FLRW metric. This leads to

$$\langle \rho_i^{(m)} \rangle = \bar{\rho}_i^{(m)}, \quad \langle \rho_i^{(r)} \rangle = \bar{\rho}_i^{(r)}, \quad \Gamma = \Psi = 1, \quad \Phi = 0,$$

where  $\bar{\rho}_i^{(m)} = mc^2 \bar{n}_i^{(m)}, \bar{\rho}_i^{(r)} = 3\bar{n}_i^{(r)} k_B \bar{T}_i$ , so that  $T = T(t), \rho = \rho(t), p = p(t)$  and  $P = 0$ , with (3) becoming a perfect fluid tensor where  $\rho$  and  $p$  satisfy (2). The FLRW limit can also be characterized by  $\Delta_i^{(m)} = \Delta_i^{(r)} = 0$ , and so, from (11), equations (21) and (22) become:  $\sigma = 0$  and  $\Theta/3 = \dot{y}/y = \dot{R}/R$ . Another limit is that of LTB dust solutions, obtained by setting  $T_i = 0$  in (12b) and (11), so that (17) becomes  $T = p = P = 0$  and  $\rho = mc^2 n^{(m)}$ .

## VI. DENSITY CONTRASTS AND REGULARITY CONDITIONS.

Since the radial dependence of all state and geometric variables is sensitive to  $\Delta_i^{(r)}$  and  $\Delta_i^{(m)}$  defined in (15), it is important to provide an interpretation for these quantities. From (14) and (15), it is evident that  $\Delta_i^{(r)}$  and  $\Delta_i^{(m)}$  are effective “gauges” of the deviation of  $\rho_i^{(m)}$ ,  $\rho_i^{(r)}$  from their volume averages for every closed interval in the range of the integration variable  $r$  along the initial hypersurface  $t = t_i$ . The signs of these quantities characterize initial density profiles, with “density lumps” as these densities decrease ( $[\rho_i^{(m)}]' < 0$ ,  $[\rho_i^{(r)}]' < 0$ ) or “density voids” as they increase ( $[\rho_i^{(m)}]' > 0$ ,  $[\rho_i^{(r)}]' > 0$ ). Also, with the help of Rolle’s theorem applied to (14) we find that  $\Delta_i^{(r)}$  and  $\Delta_i^{(m)}$ , as adimensional functions of  $r$ , are constrained by the maximal density contrasts in terms of

$$|\Delta_i^{(m)}| \leq \frac{\rho_i^{(m)\max}}{\rho_i^{(m)\min}} - 1,$$

$$|\Delta_i^{(r)}| \leq \frac{\rho_i^{(r)\max}}{\rho_i^{(r)\min}} - 1,$$

where the superindices “max” and “min” respectively indicate the maximal and minimal values of  $\rho_i^{(m)}$ ,  $\rho_i^{(r)}$  in any interval  $0 \leq r$  along the hypersurface  $t = t_i$ . Small initial density contrasts obviously imply

$$\rho_i^{(m)\max} \approx \rho_i^{(m)\min}, \quad \rho_i^{(m)} \approx \langle \rho_i^{(m)} \rangle \quad \Rightarrow \quad |\Delta_i^{(m)}| \ll 1, \quad (23a)$$

$$\rho_i^{(r)\max} \approx \rho_i^{(r)\min}, \quad \rho_i^{(r)} \approx \langle \rho_i^{(r)} \rangle \quad \Rightarrow \quad |\Delta_i^{(r)}| \ll 1, \quad (23b)$$

$$\mu \approx \frac{\kappa}{3} \rho_i^{(m)}, \quad \epsilon \approx \frac{\rho_i^{(r)}}{\rho_i^{(m)}} \approx \frac{3n_i^{(r)}}{n_i^{(m)}} \frac{k_B T_i}{mc^2} \approx 10^{-9} \frac{k_B T_i}{mc^2}, \quad (23c)$$

allowing us to consider a formal analogy between  $\Delta_i^{(r)}$  and  $\Delta_i^{(m)}$  and energy density “exact” initial perturbations. This is further reinforced from the definitions in (15), and by remarking that the FLRW “background” follows by “turning the perturbations off”, that is, setting:  $\Delta_i^{(m)} = \Delta_i^{(r)} = 0$ .

An important restriction that the solutions must satisfy is the following regularity condition

$$\Gamma \equiv \frac{Y'/Y}{Y_i'/Y_i} > 0, \quad (24)$$

which prevents negative densities  $n^{(r)}$ ,  $n^{(m)}$ , as well as the occurrence of a shell crossing singularity [35]. This singularity is characterized by unphysical behavior because  $\Gamma$  appears in the denominator of equations (17a), (17c), (17d) and (17e), but does not appear in (17b). Therefore, if  $\Gamma = 0$ , the densities, pressure and viscous pressure diverge with  $T$  finite (in general), a totally unacceptable situation that can be avoided by considering only the range of evolution of the models to spacetime sections with  $t \geq t_i$  satisfying (24). The fulfilment of (24) depends on the functions  $A, B$  and on the magnitudes of the initial density contrasts gauged by  $\Delta_i^{(r)}$  and  $\Delta_i^{(m)}$ . This will be examined further ahead together with the conditions for thermodynamical consistency.

## VII. THERMODYNAMICAL CONSISTENCY

The models derived and presented in the previous sections must be compatible with a suitable thermodynamical formalism. For this purpose, it is advisable to leave aside the “conventional” theory of irreversible thermodynamics [36], [37], whose transport equations are unphysical as they violate relativistic causality of the dissipative signals as well as stability of the equilibrium states (see.g. [18], [19] and [24]). We shall consider instead “Extended Irreversible



Thermodynamics” (EIT) [20], [21], [22], [23], a theory free of such serious drawbacks [24], and so a more adequate theoretical framework for the models. According to EIT, the entropy of a system away from thermodynamical equilibrium depends not only on the “conserved” (equilibrium) variables (i.e. particle number densities, energy density and so on), but also into the non-equilibrium fluxes (i.e. heat flux, and bulk and shear dissipative stresses). This theory is supported by Kinetic Theory of gases, Information Theory and by the Theory of Hydrodynamical Fluctuation - see [23] for a detailed description. When shear viscosity is the only dissipative agent, the corresponding generalized entropy  $s$  of radiation plus matter obeying the usual balance law with non-negative divergence, up to second order in  $\Pi^{ab}$  takes the form

$$s = s^{(e)} + \frac{\alpha}{nT} \Pi_{ab} \Pi^{ab}, \quad \Rightarrow \quad (s n u^a)_{;a} \geq 0, \quad (25)$$

where  $s^{(e)}$  is obtained from the integration of the equilibrium Gibbs equation,  $n = n^{(r)} + n^{(m)}$  and  $\alpha$  is a phenomenological coefficient to be specified later. The evolution of the viscous pressure is, in turn, governed by the transport equation

$$\tau \dot{\Pi}_{cd} h_a^c h_b^d + \Pi_{ab} \left[ 1 + \frac{1}{2} T \eta \left( \frac{\tau}{T \eta} u^c \right)_{;c} \right] + 2 \eta \sigma_{ab} = 0, \quad (26)$$

where  $\eta$  and  $\tau$  are the coefficient of shear viscosity and the relaxation time of shear viscosity, respectively. The former as well as other related quantities can be obtained by a variety of means [2] including Kinetic Theory, Statistical Mechanics or both [25], [26], [27], [28]. The relaxation time,  $\tau$ , is related to and larger than the mean collision time between particles and it may, in principle, be estimated by collision integrals provided the interaction potential is known. As a physical reference to infer the form these coefficients might take, consider the “radiative gas”, with  $p$ ,  $\rho$  satisfying (1) (or the approximation (2)). For the radiative gas the forms of  $\eta$ ,  $\alpha$  in terms of the relaxation time of the dissipative process  $\tau$  are

$$\eta_{(rg)} = \frac{4}{5} p^{(r)} \tau, \quad \alpha_{(rg)} = -\frac{\tau}{2 \eta_{(rg)}} = -\frac{5}{8 p^{(r)}} \quad (27)$$

where  $p^{(r)}$  is either  $n^{(r)} k_B T$  or  $a_B T^4$  and the subscript “(rg)” emphasizes that these quantities are specific to the radiative gas.

We verify now the compatibility of (13)-(22) with (25)-(27). Integrating the equilibrium Gibbs equation and substituting (27) into (25), we obtain

$$\begin{aligned} s &= \frac{4 a_B T_i^3}{3 n_i^{(r)}} + k_B \ln \left[ \frac{n_i^{(r)}}{n^{(r)}} \left( \frac{T}{T_i} \right)^3 \right] - \frac{15 k}{8} \left( \frac{P}{p} \right)^2 \\ &= \frac{4 a_B T_i^3}{3 n_i^{(r)}} + k_B \ln \left[ \frac{\Psi^3}{\Gamma} \right] - \frac{15 k}{32} \left( \frac{\Phi}{\Psi} \right)^2, \end{aligned} \quad (28)$$

where the approximation:  $n = n^{(m)} + n^{(r)} \approx n^{(r)}$  was used and the initial value of  $s^{(e)}$  has been set to be the equilibrium entropy per photon. Equation (28) reflects the fact that we are neglecting the contribution of the entropy due to non-relativistic particles, a justified approximation since the latter are much less abundant than the photons.

Ideally, the transport equation (26) should be satisfied for  $\eta$  having the form (27), associated to the radiative gas, and the relaxation time,  $\tau$ , given by collision integrals obtained from Kinetic Theory. However, as mentioned in the introduction, and in order to obtain exact expressions for all thermodynamical parameters, we will assume  $\eta$  given by (27) and deduce  $\tau$  from the fulfilment of (26). This yields

$$\tau = \frac{-\Psi \Phi}{\sigma} \frac{\frac{9}{4} \left[ 1 + \Delta_i^{(r)} \right]^2}{\frac{4}{5} \left[ 3 + 4 \Delta_i^{(r)} + \frac{13}{32} \Gamma \right]^2 + \frac{171}{256} \Gamma^2}. \quad (29)$$

While there is no need to justify  $\eta$  given by (27), this form of  $\tau$  is acceptable as long as (29) satisfies the requirement of a relaxation parameter: it must be a positive quantity and must comply with a positive entropy production law  $\dot{s} \geq 0$ . It is desirable that  $\tau$  should somehow relate or approach its definition as a collision integral and that its behavior be qualitatively analogous to a suitable mean collision time, therefore it should be an increasing (decreasing) function if the fluid is expanding (collapsing). Evaluating  $\dot{s}$  from (28) and comparing with (29), we obtain the following relation between  $\dot{s}$  and  $\tau$

$$\dot{s} = \frac{15k_B}{4\tau} \left( \frac{P}{p} \right)^2 = \frac{15k_B}{16\tau} \left( \frac{\Phi}{\Psi} \right)^2, \quad (30)$$

consistent with the general relation [22]  $(nsu^a)_{;a} = \Pi_{ab}\Pi^{ab}/(2\eta nT)$  associated with (25) and (26). As a consequence of (30),  $\dot{s} > 0$  and  $\tau > 0$  imply each other. Also, from the form of (29), necessary and sufficient conditions for positive  $\tau$ ,  $\dot{s}$ ,  $p$ ,  $T$  can be given by

$$\Psi > 0, \quad (31a)$$

$$\sigma\Phi < 0, \quad (31b)$$

while the condition ensuring concavity and stability of  $s$  can be phrased for an expanding fluid configuration as the requirement that  $\dot{s}$  decreases for increasing  $\tau$  ( $\dot{\tau} > 0 \Leftrightarrow \dot{s} < 0$ ). From (30)-(31), this follows as

$$\dot{\tau} > 0, \quad \frac{\ddot{s}}{\dot{s}} = \frac{2\sigma\Gamma}{3\Psi\Phi} \frac{\langle \rho_i^{(r)} \rangle}{\rho_i^{(r)}} \left[ 1 + \frac{\langle \rho_i^{(r)} \rangle}{3\rho_i^{(r)}} \right] - \frac{\dot{\tau}}{\tau} < 0. \quad (31c)$$

So that, if (24), (31a) and (31b) hold, then (31c) reduces to  $\dot{\tau} > 0$ .

Since  $\tau$  is a thermodynamic relaxation parameter, it is important to compare it with another natural timescale of the models: the Hubble expansion time defined by

$$t_H = \frac{3}{\Theta}. \quad (32a)$$

where  $\Theta$  follows from (22). Such comparison should provide an insight into the timescales associated with the mixture interaction and decoupling. However, strictly speaking, the criterion for interaction and decoupling in cosmological gas mixtures is not given by comparing  $\tau$  and  $t_H$  but by comparing the latter with the timescales associated with the various reaction rates of the radiative processes involved, particularly the photon mean collision time  $t_\gamma$  obtained from Thomson scattering [5], [6]. Hence, we can consider this relaxation time as approximately gauging the interactivity of the matter mixture by demanding that for a range of the evolution of the mixture, approximating its interactive range, we must have

$$\tau < t_H \quad (32b)$$

while, as the mixture evolves and the components decouple, eventually

$$\tau > t_H. \quad (32c)$$

Since  $\tau$  in (27) and (29) must behave qualitatively similar to  $t_\gamma$  [29], the comparison in (32b-c) should yield qualitatively analogous results as a similar comparison between  $t_H$  and  $t_\gamma$ . The temperature associated with the passage from (32b) to (32c) (obtained from (17b)) should approximate the decoupling temperature obtained by the condition  $t_\gamma = t_H$ . This point is examined in section XI.

Equations (31)-(32), together with the regularity condition (24), provide the necessary and sufficient conditions for a theoretically consistent thermodynamical description of the solutions within the framework of EIT and Kinetic Theory applied to the radiative gas. We examine the effect of these conditions in the following section.

## VIII. THERMODYNAMICALLY CONSISTENT MODELS

### A. Conditions (24) and (31a)

From (19a) and (19b), the fulfilment of (24) and (31a) is equivalent to the following condition

$$-\frac{1}{3} < A\Delta_i^{(m)} + B\Delta_i^{(r)} < 1 + \Delta_i^{(r)}. \quad (33)$$

From (20), since the functions  $A$  and  $B$  diverge as  $y \rightarrow 0$  there are necessarily values of  $y$  for which (33) is violated. However, from (19a-b) we have:  $\Gamma_i = \Psi_i = 1$ , and the range of  $y$  we are interested is  $y \geq 1$ . As shown by figures 1a and 1b, displaying the implicit plots of  $\Gamma = 0$  and  $\Psi = 0$  for an initial temperature  $T_i \approx 10^6$  K (from (23c),  $\epsilon \approx 10^3$ ), there are no zeros of  $\Gamma$  and  $\Psi$  for  $y \geq 1$ ,  $|\Delta_i^{(m)}| \leq 1$  and  $|\Delta_i^{(r)}| \leq 1$ . Therefore, (33) holds in the range of interest for a large class of initial conditions.

### B. Conditions (31b), (31c) and (32)

For the examination of these conditions we will assume that (33) holds for  $y \geq 1$ ,  $|\Delta_i^{(m)}| \leq 1$  and  $|\Delta_i^{(r)}| \leq 1$  (see figures 1a and 1b). Then, with the help of (19)-(22), condition (31b) is equivalent to

$$C \equiv 4AA_{,y} [\Delta_i^{(m)}]^2 + (4B - 1)B_{,y} [\Delta_i^{(r)}]^2 + [4AB_{,y} + (4B - 1)A_{,y}] \Delta_i^{(m)} \Delta_i^{(r)} < 0. \quad (34a)$$

An insight into this expression follows by looking at its initial value

$$C_i = \frac{-\Delta_i^{(r)}}{2(1 + \epsilon)} [\Delta_i^{(r)} + \epsilon \Delta_i^{(m)}], \quad (34b)$$

and its asymptotical behavior as  $y \gg 1$

$$\begin{aligned} C \approx & \frac{4\epsilon [\Delta_i^{(s)}]^2}{9y^2} + \frac{4\lambda \Delta_i^{(s)}}{\sqrt{1 + \epsilon} y^{5/2}} + \frac{35}{9} \frac{\epsilon \lambda \Delta_i^{(s)}}{\sqrt{1 + \epsilon} y^{7/2}} \\ & - \frac{\lambda_1 [\Delta_i^{(s)}]^2 + 24\lambda_2 \Delta_i^{(r)} \Delta_i^{(s)} + 9 [\Delta_i^{(r)}]^2}{24(1 + \epsilon) y^4} \\ & + \frac{64}{63} \frac{\epsilon^2 \lambda \Delta_i^{(s)}}{\sqrt{1 + \epsilon} y^{9/2}} - \frac{\epsilon}{18} \frac{\lambda_1 [\Delta_i^{(s)}]^2 + 24\lambda_2 \Delta_i^{(r)} \Delta_i^{(s)} + 9 [\Delta_i^{(r)}]^2}{(1 + \epsilon) y^4} \\ & - \frac{55}{8} \frac{\epsilon^3 \lambda \Delta_i^{(s)}}{\sqrt{1 + \epsilon} y^{11/2}} + \mathcal{O}(y^{-6}), \end{aligned} \quad (34c)$$

where

$$\Delta_i^{(s)} \equiv \frac{3}{4} \Delta_i^{(r)} - \Delta_i^{(m)}, \quad (35)$$

and

$$\lambda \equiv \Delta_i^{(m)} \lambda_2 - 3\epsilon(1 + 2\epsilon) \Delta_i^{(r)},$$

$$\lambda_1 \equiv 1 - 8\epsilon + 2\epsilon^3 + 2\epsilon^4, \quad \lambda_2 \equiv 8\epsilon^2 + 4\epsilon - 1.$$

As shown by (34c), the quantity  $\Delta_i^{(s)}$  defined in (35) plays a fundamental role with regards to the fulfilment of (31b) and (31c). This suggest classifying initial conditions in terms of  $\Delta_i^{(s)}$ . We shall examine (31b), (31c) and (32) for each case  $\Delta_i^{(s)} = 0$  and  $\Delta_i^{(s)} \neq 0$  separately.

1. The case  $\Delta_i^{(s)} = 0$

From (34c), the condition  $\Delta_i^{(s)} = 0$  is necessary and sufficient for having  $C < 0$  asymptotically. In fact, substituting  $\Delta_i^{(s)} = 0$  into (34a), leads to the following dramatic simplification of  $C$

$$C = -\frac{\left[\Delta_i^{(r)}\right]^2 (3y + 4\epsilon)}{8(1 + \epsilon)y^5}, \quad (36)$$

so that  $\Delta_i^{(s)} = 0$  is a sufficient condition for the fulfillment of (24), (31a) and (31b) along the range  $y \geq 1$ , for  $|\Delta_i^{(r)}| \leq 1$  and  $|\Delta_i^{(m)}| < 1$ .

Condition (31c) is also satisfied, since  $\tau$  increases monotonously along the fluid worldlines, behaving asymptotically as:  $\tau \approx y^{3/2}$ . Hence, those models whose initial conditions satisfy  $\Delta_i^{(s)} = 0$  can be characterized as the subclass of models complying with the conditions of thermodynamical consistency in the asymptotic range of  $y$ . This point is consistent with the fact that  $\Delta_i^{(s)} = 0$  is a necessary and sufficient condition for having  $|P|/p = |\Phi|/(2\Psi) \rightarrow 0$  and  $\dot{s} \rightarrow 0$  as  $y \rightarrow \infty$ , so that the fluid layers evolve towards an asymptotic equilibrium state. However, using  $t_H = 3/\Theta$  calculated by inserting  $\Delta_i^{(s)} = 0$  into (22), we have  $\tau/t_H < 1$  for all the evolution of the fluid, thus failing to comply with (32b-c). This is illustrated by figure 2, and implies that the relaxation time  $\tau$  cannot be associated with a radiation matter mixture whose components interact and then decouple.

2. The case  $\Delta_i^{(s)} \neq 0$

If  $\Delta_i^{(s)} \neq 0$ , condition (31b) cannot hold along the full range of  $y$  (because of (34b)), but might hold along a restricted range of physical interest  $1 \leq y \leq y_*$  for which the mixture could be in the interactive stage. This situation is not incompatible with the thermodynamical arguments of the previous section, since the phenomenology of the radiative gas model strictly applies if the mixture components interact. The fact that condition (31b) can hold for  $1 \leq y \leq y_*$  follows from evaluating the sign of  $C_i$  given by (34a), a quantity that can be negative (so that  $\tau_i > 0$ ) for a wide range of acceptable situations (for example, if  $\Delta_i^{(r)}$  and  $\Delta_i^{(m)}$  have the same sign). However, as  $y$  increases,  $\tau$  either changes sign or diverges positively, depending on the zeroes of  $\Phi$  and  $\sigma$ . Since we are assuming that (24) and (31a) hold, the sign of  $\tau$ , as given by (29), depends only on the quotient  $\Phi/\sigma$ , and so the behavior of  $\tau$  is strongly related to the signs and zeroes of these functions. If  $\tau_i > 0$  and, as  $y$  increases, there is a zero of  $\Phi$  for  $\sigma \neq 0$ , then  $\tau$  passes from positive to negative, but if the zero of  $\sigma$  appears first, then  $\tau$  diverges positively. A zero of  $\Phi$  (with  $\sigma \neq 0$ ) might be compatible with (31b) ( $\tau > 0$ ), but violates (31c) and so is unacceptable. However, a zero of  $\sigma$  (with  $\Phi \neq 0$ ) is acceptable, since  $\tau$  would diverge positively (as  $y \rightarrow y_*$ ) and so would be a positive and increasing function along the range  $1 \leq y \leq y_*$ . In order to verify if this type of evolution is possible, it is necessary to gather information on the zeroes of  $\Phi$  and  $\sigma$ . Since these expressions are cumbersome, it is convenient to examine their zeros graphically, and so we have plotted in figures 3a and 3b the solutions of the implicit equations  $\Phi = 0$  and  $\sigma = 0$ , in terms of  $\Delta_i^{(m)}$ ,  $\Delta_i^{(r)}$  and  $\log_{10}(y)$ , while figure 4 displays the sectors in the plane  $\Delta_i^{(m)}$ ,  $\Delta_i^{(r)}$  where  $\Phi = 0$  and  $\sigma = 0$  occur. As these figures reveal, sufficient conditions for the desired evolution are given by

$$\Delta_i^{(s)} > 0 \quad \text{for} \quad \Delta_i^{(r)} > 0 \quad (37a)$$

$$\Delta_i^{(s)} < 0 \quad \text{for} \quad \Delta_i^{(r)} < 0. \quad (37b)$$

Therefore, if  $\Delta_i^{(s)} \neq 0$ , conditions (37) are sufficient for the fulfillment of (31b) and (31c) along the range  $1 \leq y \leq y_*$ , where  $y_*$  is a zero of  $\sigma$ . Conditions (32) are satisfied under the restrictions (37). This will be discussed further ahead in section X.

## IX. INITIAL CONDITIONS AND EXACT INITIAL PERTURBATIONS.

The behavior of the quantity  $\Delta_i^{(s)}$  given by (35) determines the set of initial conditions that characterize the thermodynamical consistency of the models. Since this quantity plays such an important role, it must be related to a physically significant property along the initial hypersurface  $t = t_i$ . From (35), using (12), (14) and (15), we obtain

$$\Delta_i^{(s)} = \frac{3}{4}\Delta_i^{(r)} - \Delta_i^{(m)} = \frac{1}{4} + \frac{[\log(W^{3/4}/M)]'}{[\log(Y_i^3)]'} = \frac{d[\log \Sigma_i]}{d[\log Y_i^3]}, \quad (38a)$$

where

$$\Sigma_i \equiv \frac{\langle \rho_i^{(r)} \rangle^{3/4}}{\langle \rho_i^{(m)} \rangle}. \quad (38b)$$

In order to provide an interpretation for (38), we remark (from (17), (19) and (28)) that the entropy per photon along the initial hypersurface

$$s_i = \frac{4a_B T_i^3}{3n_0^{(r)}} - \frac{15}{32}k_B \left[ \frac{\Phi_i}{\Psi_i} \right]^2 = \frac{4a_B T_i^3}{3n_i^{(r)}} - \frac{15}{32}k_B \left[ \frac{\Delta_i^{(r)}}{1 + \Delta_i^{(r)}} \right]^2$$

is very close to its equilibrium value, since its off equilibrium correction is proportional to  $[\Delta_i^{(r)}]^2$ , a very small quantity since  $\Delta_i^{(r)}$  is already assumed to be small. Therefore, using the approximations associated with small density contrasts (equations (23a-c)), we obtain

$$\rho_i^{(r)} \approx 3a_B T_i^4, \quad \Sigma_0 \propto \frac{\langle a_B T_i^4 \rangle^{3/4}}{mc^2 \langle n_i^{(m)} \rangle} \propto \frac{\langle s_i \rangle}{\langle n_i^{(m)} \rangle} \approx \left\langle \frac{s_i}{n_i^{(m)}} \right\rangle, \quad (39)$$

implying that  $\Sigma_i$  is proportional to the ratio of the averages of photon entropy and baryon number density, which (for small density contrasts) is roughly equivalent to the averaged ratio of these quantities. Hence, condition  $\Delta_i^{(s)} = 0$  roughly means an initial hypersurface with constant averages of photon entropy per baryon, while condition (37) roughly means (see equations (20)) that the sign of the spatial gradient of the average of photon entropy per baryon must agree with the sign of the gradient of the initial photon energy density: as  $r$  increases along  $t = t_i$ , it must increase for a density void ( $[\rho_i^{(r)}]' > 0$ ) and decrease for a density lump ( $[\rho_i^{(r)}]' < 0$ ).

The assumption of small density contrasts leads to a natural comparison with the Theory of Perturbations of a FLRW background, in the isochronous gauge and considering a mixture of radiation and non-relativistic matter, see [4], [5], [6], [8], [9], [10]. Under these assumptions, matter and radiation densities are given by

$$\rho^{(m)} = \bar{\rho}^{(m)} \left[ 1 + \delta^{(m)} \right], \quad (40a)$$

$$\rho^{(r)} = \bar{\rho}^{(r)} \left[ 1 + \delta^{(r)} \right], \quad (40b)$$

where  $\bar{\rho}^{(m)}$ ,  $\bar{\rho}^{(r)}$  are the respective densities in a FLRW background and  $\delta^{(m)} \ll 1$ ,  $\delta^{(r)} \ll 1$  are the perturbations. The gauge invariant quantity

$$\delta^{(s)} = \frac{3}{4}\delta^{(r)} - \delta^{(m)}, \quad (41a)$$

which formally resembles (35), defines the fluctuations of photon entropy per baryon and leads to the classification of perturbations as

$$\text{“Adiabatic”} \quad \delta^{(s)} = 0, \quad (41b)$$

$$\text{“Isocurvature”} \quad \delta^{(s)} = \delta^{(m)} \left[ 1 + \frac{3\rho^{(m)}}{4\rho^{(r)}} \right]. \quad (41c)$$

In order to establish a comparison with Perturbation Theory, consider defining the initial densities along the lines of (40)

$$\rho_i^{(m)} = \bar{\rho}_i^{(m)} \left[ 1 + \delta_i^{(m)} \right] \Rightarrow \langle \rho_i^{(m)} \rangle = \bar{\rho}_i^{(m)} \left[ 1 + \langle \delta_i^{(m)} \rangle \right], \quad (42a)$$

$$\rho_i^{(r)} = \bar{\rho}_i^{(r)} \left[ 1 + \delta_i^{(r)} \right] \Rightarrow \langle \rho_i^{(r)} \rangle = \bar{\rho}_i^{(r)} \left[ 1 + \langle \delta_i^{(r)} \rangle \right], \quad (42b)$$

where  $\bar{\rho}_i^{(m)}, \bar{\rho}_i^{(r)}$  are now the constant values of the initial densities in a FLRW background and  $|\delta_i^{(m)}| \ll 1, |\delta_i^{(r)}| \ll 1$  are exact initial perturbations. Inserting (42) into (38) leads to

$$\Sigma_i \approx \frac{\bar{s}_i}{\bar{n}_i^{(m)}} \frac{1 + \frac{3}{4} \langle \delta_i^{(r)} \rangle}{1 + \langle \delta_i^{(m)} \rangle} \Rightarrow \frac{\Delta_i^{(s)}}{Y_i^3} \approx \frac{d}{dY_i^3} \left\langle \frac{3}{4} \delta_i^{(r)} - \delta_i^{(m)} \right\rangle \approx \frac{d}{dY_i^3} \langle \delta_i^{(s)} \rangle, \quad (42c)$$

where  $\bar{s}_i \propto [\bar{\rho}_i^{(r)}]^{3/4}$  and  $\bar{n}_i^{(r)}$  are the photon entropy and baryon number density of the FLRW background in the hypersurface  $t = t_i$ . The rhs of (42c) illustrates that, under the assumptions (42a-b) of a perturbative treatment,  $\Delta_i^{(s)}$  approximately reduces to the radial gradient of  $\delta^{(s)}$  evaluated in  $t = t_i$ . However, the comparison with perturbations must be handled with caution, since in the case of (35) and (36)-(39) we are dealing with quantities and relations strictly defined in the initial hypersurface, and so determining initial conditions in terms of averages of initial value functions. The Theory of Perturbations, on the other hand, traces the evolution of  $\delta^{(m)}, \delta^{(r)}$  for all  $t$ . In particular, we must be specially careful if we borrow the terminology of perturbations in (41) to characterize the cases  $\Delta_i^{(s)} = 0$  and  $\Delta_i^{(s)} \neq 0$ , since in our case a relation like  $\Delta_i^{(s)} = 0$  specifies initial conditions and will not be satisfied (in general) for  $t > t_i$ . Another delicate point refers to the current use of the term “adiabatic” for the case (41b), meaning perturbations that conserve photon entropy. Strictly speaking, these perturbations should be denoted as “isentropic” or “reversible”, since an adiabatic process need not be isentropic (or reversible). It is quite possible that this conceptual vagueness follows from the fact that most papers in Perturbation Theory, either assume thermal equilibrium, or have incorporated dissipative processes without the necessary rigour. Hopefully, recent work [17] on these lines might be helpful to clarify these issues. Raising this point is relevant because the models presented in this paper assume a fluid evolving along adiabatic but irreversible (non-isentropic) processes, therefore initial conditions like  $\Delta_i^{(s)} = 0$  (formally analogous to (41b)) do not imply a conserved photon entropy, but a roughly constant average of photon entropy at  $t = t_i$ , but not (in general) for  $t > t_i$ . Never the less, for the sake of maintaining continuity with currently established terminology, we shall use the formal analogy between (41) and (42) in the examination of the cases  $\Delta_i^{(s)} = 0$  and  $\Delta_i^{(s)} \neq 0$ . Developing further the comparison with perturbations on a FLRW background is relevant and interesting. However, such a task requires a comprehensive and detailed elaboration and so will be carried on elsewhere.

## X. THE CASE $\Delta_I^{(S)} \neq 0$ , AN INTERACTIVE MIXTURE.

From section VIII we know that if (37a-b) hold, then  $\tau$  diverges positively as  $y \rightarrow y_*$  where  $y_*$  is a zero of  $\sigma$ . We still need to know if  $\tau/t_H < 1$  along the range  $1 \leq y < \tilde{y}$ , where the set of values  $y = \tilde{y} < y_*$  are characterized by  $\tau/t_H = 1$ , approximately marking the decoupling of matter and radiation. A sufficient condition for this type of evolution follows from evaluating  $[\tau/t_H]_i$  at the initial hypersurface

$$\left[ \frac{\tau}{t_H} \right]_i = \frac{5\Delta_i^{(r)} \left[ 1 + \Delta_i^{(r)} \right] \left[ (3 + 4\epsilon)\Delta_i^{(r)} - 4\Delta_i^{(s)} + 8(1 + \epsilon) \right]}{\left[ (3 + 4\epsilon)\Delta_i^{(r)} - 4\Delta_i^{(s)}(3 + 4) \right] \left[ 36 + 47\Delta_i^{(r)} + 16 \left( \Delta_i^{(s)} \right)^2 \right]} \approx \frac{10(1 + \epsilon)}{9(3 + 4\epsilon)} \approx \frac{5}{18} < 1,$$

where we have eliminated  $\Delta_i^{(m)}$  from (35) and assumed  $\Delta_i^{(s)} \ll 1, \Delta_i^{(r)} \ll 1$  and  $\epsilon \approx 10^{-3}T_i \approx 10^3$ . Since  $\tau$  diverges as  $y \rightarrow y_*$  but  $\Theta$  remains finite in this limit, the ratio  $\tau/t_H$ , initially smaller, necessarily becomes larger than unity for  $1 < y < \tilde{y} < y_*$ .

The currently accepted value of matter and radiation decoupling is  $T_D \approx 4 \times 10^3$  K. Assuming  $T_i \approx 10^6$  K, the set of values  $y = y_D$  associated with this temperature follow from (17b) by solving for  $y = y_D$  the equation

$$T_D \approx 4 \times 10^3 = \frac{10^6}{y_D} \Psi(y_D, \Delta_i^{(s)}, \Delta_i^{(r)}), \quad (43)$$

where  $\Psi$  is given by (19b). Assuming small density contrasts, figure 5 illustrates that this value of  $T_D$  is closely associated with  $y_D \approx 10^{2.4}$ . Also, it is evident from figure 3b, that having a zero of  $\sigma$  for values  $10^2 < y_* < 10^3$  requires a very small deviation from  $\Delta_i^{(s)} = 0$ . This is illustrated by the approximated sketch of the level curve  $y = y_D = 10^{2.4}$  that appears in figure 4, implying that  $\tilde{y}$ ,  $y_*$  and  $y_D$  lie in a very narrow sector of the plane  $\Delta_i^{(m)}, \Delta_i^{(r)}$ , very close to the line  $\Delta_i^{(s)} = 0$ . Therefore, for these values we must have  $y_* \approx \tilde{y} \approx y_D$ . However, it is not clear from figures 3b and 4 how small the deviation from  $\Delta_i^{(s)} = 0$  should be. Hence, we have plotted in figure 6 the implicit equation  $\sigma = 0$ , showing a more precise relation between the orders of magnitude that the occurrence of the zero of  $\sigma$  for  $y_D \approx y_* \approx 10^{2.4}$  implies for  $|\Delta_i^{(r)}|$ ,  $|\Delta_i^{(s)}|$ .

Another constraint the models have to comply with is the observational bounds [5], [6] on the anisotropy of the cosmic microwave radiation (CMR), given by the maximal photon temperature contrast  $[\delta T/T]_D \approx 10^{-5}$ . For  $y \approx 10^{2.4}$  and  $T_i \approx 10^6$ , so that  $\epsilon \approx 10^3$ , we have:  $A \approx 1.1 \times 10^{-1}$  and  $B \approx 2.2 \times 10^{-1}$ , were  $A$  and  $B$  are given by (20). Also, from (17b) and (19b), the deviation from the equilibrium FLRW form  $T_{eq} = T_i/y$  at  $y = y_D$  is given by

$$\left[ \frac{\delta T_{eq}}{T_{eq}} \right]_D = \left[ \frac{T - T_{eq}}{T_{eq}} \right]_D = [\Psi - 1]_D \approx -\frac{1.1 \times 10^{-1} \Delta_i^{(m)} + 2.2 \times 10^{-1} \Delta_i^{(r)}}{1 + \Delta_i^{(r)}}, \quad (44)$$

indicating that compliance with the maximal observed temperature contrast of the CMR constrains the maximal values of  $\Delta_i^{(m)}$ ,  $\Delta_i^{(r)}$  to about  $10^{-4}$ . Therefore, from figure 6, the corresponding variation range of  $|\Delta_i^{(s)}|$  is  $|\Delta_i^{(s)}| < 10^{-8}$ , and so compatibility with acceptable values of  $T_D$  and the CMR anisotropy implies  $|\Delta_i^{(s)}| \approx |\Delta_i^{(r)}|^2 \ll |\Delta_i^{(r)}|$ .

Under the analogy between  $\Delta_i^{(m)}$ ,  $\Delta_i^{(r)}$  with perturbations in a FLRW background, the case  $\Delta_i^{(s)} \neq 0$  seems to correspond to isocurvature initial perturbations. However, since  $\rho^{(m)} \ll \rho^{(r)}$  before decoupling, the latter are characterized (see equation (41c)) by  $\delta^{(s)} \approx \delta^{(m)}$  and  $\delta^{(r)} \ll \delta^{(s)}$ , and so, following the analogy between (41) and (42) would disqualify  $\Delta_i^{(s)} \neq 0$  as comparable to an isocurvature initial perturbation. In fact, the maximal bounds on  $\Delta_i^{(r)}$  and  $\Delta_i^{(s)}$  obtained in the previous paragraph yield exactly the opposite behavior. Borrowing the terminology of Perturbation Theory, the analogy between (41) and (42) would characterize “isocurvature” initial conditions by  $\Delta_i^{(r)} = 0$ . From figures 3 and 4, this condition is not incompatible with (31) and (32), but yields a decoupling surface  $y_D \approx 10$  with a decoupling temperature much larger than the accepted value. Hence, the acceptable values of the case  $\Delta_i^{(s)} \neq 0$  cannot be associated with this type of initial perturbations, but since perturbations are in general combinations of adiabatic and isocurvature components and we have  $\Delta_i^{(s)} \ll \Delta_i^{(r)}$  and  $\Delta_i^{(s)} \ll \Delta_i^{(m)}$ , a more accurate analogy for  $\Delta_i^{(m)}$ ,  $\Delta_i^{(r)}$  is that of “quasi adiabatic” initial perturbations. Following this analogy, and bearing in mind that  $\Delta_i^{(m)}$  and  $\Delta_i^{(r)}$  are defined for an initial hypersurface characterized by  $T_i \approx 10^6$  K. (still outside the horizon for perturbations of all wave numbers), the bounds on the magnitude of  $\Delta_i^{(s)}$  can be related to bounds on the amplitude of “nearly adiabatic” fluctuations of photon entropy per baryon generated in the earlier, more primordial, inflationary era [4], [5], [6], [10], [34]. Primordial entropy fluctuations have been examined in connection with the creation of axions in an inflationary scenario [5], [10]. However, these are isocurvature perturbations, and so might not be related to entropy fluctuations associated with  $\Delta_i^{(s)} \neq 0$ . The study of entropy fluctuations in inflationary models is still a highly speculative topic [34], and its connection with initial conditions in the models under consideration should be an exciting subject to examine in a future paper.

## XI. SAHA’S EQUATION AND THE JEANS MASS

As mentioned earlier, it is important to compare  $\tau$  and  $t_H$  with the timescale characteristic of the interaction rate of the photons and electrons for Compton and Thomson scattering (the dominant radiative processes in the temperature range of interest)

$$t_\gamma = \frac{1}{c\sigma_T n_e}, \quad t_c = \frac{m_e c^2}{k_B T} t_\gamma, \quad (45)$$

where  $\sigma_T \approx 6.65 \times 10^{-25} \text{cm}^2$  is the Thomson scattering cross section,  $m_e$  is the electron mass and  $n_e$  is the number density of free electrons, a quantity obtained from Saha's equation

$$\frac{X_e^2}{1 - X_e} = \left[ \frac{2\pi m_e k_B T}{h^2} \right]^{3/2} \frac{\exp(B_0/k_B T)}{n_B}, \quad (46)$$

$$X_e \equiv \frac{n_e}{n_B}, \quad n_B \approx n^{(m)},$$

where  $X_e$  is the fractional ionization,  $h$  is Planck's constant,  $n_B$  is the number density of baryons and  $B_0 \approx 13.6 \text{ eV}$  is the binding energy of the hydrogen atom. Combining (45) and (46) we obtain

$$t_\gamma = \frac{1}{2c\sigma_B n^{(m)}} \left[ 1 + \left( 1 + \frac{4h^3 n^{(m)} \exp(B_0/k_B T)}{(2\pi m_e k_B T)^{3/2}} \right)^{1/2} \right]. \quad (47)$$

The recombination process is characterized by  $X_e \approx 1/10$  in (46), so that most free electrons have combined with protons into neutral atoms, while the decoupling of matter and radiation strictly follows from the condition:  $t_\gamma = t_H$ , a condition analogous to  $\tau = t_H$  and leading to the “decoupling temperature” from (45) and (47) with the help of (17b) and (22). Since  $t_\gamma$  must be smaller but qualitatively similar to  $\tau$  (increasing as  $\tau$  increases [29]), a comparison between  $\tau$  and  $t_H$  for near decoupling temperatures should be qualitatively analogous to that between  $t_\gamma$  and  $t_H$ . The timescales  $t_\gamma$  and  $t_c$  can be easily obtained from (45), (47) and (22) and compared with  $t_H$ ,  $\tau$  for a set of parameters characteristic of thermodynamically consistent models discussed in the previous section. As shown by figure 7, displaying the ratios of  $\tau, t_\gamma, t_c$  to  $t_H$  for  $\Delta_i^{(r)} = 10^{-4}$  and  $10^{-8} < \Delta_i^{(s)} < 10^{-4}$ . For  $\Delta_i^{(s)} \approx 10^{-8}$  and for near decoupling temperatures ( $10^3 < T < 10^4 \text{ K}$ . or equivalently  $10^2 < y < 10^{2.6}$ ) we have  $t_\gamma$  smaller than  $\tau$  (but of the order of magnitude) and overtaking  $t_H$  along a set of values of  $y$  that closely match the accepted value  $T_D \approx 4 \times 10^3 \text{ K}$ . This is consistent with the estimation:  $\Delta_i^{(s)} \approx 10^{-8}$  obtained in the previous section (see figure 6). On the other hand, for higher temperatures ( $T \approx 10^6 \text{ K}$ . or equivalently  $y \approx 1$ ), the relaxation parameter  $\tau$  is of the order of magnitude but greater than the Compton timescale, the dominant radiative process at these temperatures. It is also possible to show that  $X_e \approx 1/10$  in (46) leads to the accepted values of the recombination temperature.

To end the discussion, we compute the Jeans mass associated to the initial conditions of the case  $\Delta_i^{(s)} \neq 0$ . This mass is given by [1], [3], [4], [5], [6]

$$M_J = \frac{4\pi}{3} m n^{(m)} \left[ \frac{8\pi^2 C_s^2}{\kappa(\rho + p)} \right]^{3/2} = \frac{4\pi}{3} \frac{c^4 \chi_0 \Gamma^{1/2}}{\sqrt{\rho_i^{(r)}}} \left[ \frac{\pi y^2 \Psi}{3G (\Psi + \frac{3}{4} \chi_i y)^2} \right]^{3/2}, \quad (48)$$

where  $\rho, p, n^{(m)}$  are given by (2), (7), (13) and (14),  $\chi_i = \rho_i^m / \rho_i^r$ ,  $\Psi$  and  $\Gamma$  follow from (19) and  $C_s$  is the speed of sound, which for the equation of state (2), has the form

$$C_s^2 = \frac{c^2}{3} \left[ 1 + \frac{3\rho^{(m)}}{4\rho^{(r)}} \right]^{-1}, \quad \rho^{(m)} = mc^2 n^{(m)}, \quad \rho^{(r)} = 3n^{(r)} k_B T.$$

Evaluating (48) for  $y = y_D \approx 10^{2.4}$ ,  $\epsilon \approx 1/\chi_i \approx 10^3$  and  $\rho_i^{(r)} \approx a_B T_i^4 \approx 7.5 \times 10^9 \text{ ergs/cm}^3$ , yields  $M_J \approx 10^{49} \text{ gm}$ , or approximately  $10^{16}$  solar masses. This value coincides with the Jeans mass obtained for baryon dominated perturbative models as decoupling is approached in the radiative era.



## XII. CONCLUSIONS

We have derived a new class of exact solutions of Einstein's equations providing a physically plausible hydrodynamic description of a radiation-matter (photon-baryon) interacting mixture, evolving along adiabatic but irreversible thermodynamic processes. The conditions for these models to be consistent with the transport equation and entropy balance law of EIT (when shear viscosity is the only dissipative agent) have been provided explicitly, and their effect on initial conditions have been given in detail, briefly discussing the analogy of these conditions with gauge invariant initial perturbations in the isochronous gauge. As far as we are aware, and in spite of their limitations mentioned in the introduction, we believe these models are the first example in the literature of a self consistent hydrodynamical approach to matter-radiation mixtures that: (a) is based on inhomogeneous exact solutions of Einstein's field equations, and (b) is thermodynamically consistent. We believe these models can be a useful theoretical tool in the study of cosmological matter sources, providing a needed alternative and complement to the usual approaches based on perturbations or on numerical methods.

The solutions have an enormous potential as models in applications of astrophysical and cosmological interest. Consider, for example, the following possibilities: (a) *Structure formation in the acoustic phase*. There is a large body of literature on the study of acoustic perturbations in relation to the Jeans mass of surviving cosmological condensations. Equations of state analogous to (2) are often suggested in this context [1], [2], [3], [4], [6], [10]. Since practically all work on this topic has been carried on with perturbations on a FLRW background, the exact solutions derived and presented here may be viewed as an alternative treatment for this problem. (b) *Comparison with Perturbation Theory*. The models presented in this paper are based on exact solutions of Einstein's field equations, but their initial conditions and evolution can be adapted for a description of "exact spherical perturbations" on a FLRW background. It would be extremely interesting, not only to compare the results of this approach with those of a perturbative treatment, but to provide a physically plausible theoretical framework to examine carefully how much information is lost in the non-linear regime that falls beyond the scope of linear perturbations. We have studied in this paper only the case  $F = 0$  in (11), thus restricting the evolution to the "growing mode" since all fluid layers expand monotonously. The study of the more general case, where  $F(r)$  in (11) is an arbitrary function that could change sign, would allow a comparison with perturbations that include also a "decaying mode" related to condensation and collapse of cosmological inhomogeneities. (c) *Inhomogeneity and irreversibility in primordial density perturbations*. The initial conditions of the models with  $\Delta_i^{(s)} \neq 0$  are set for a hypersurface with temperature  $T_i \approx 10^6$ . These initial conditions can be considered the end product of processes characteristic of previous cosmological history, and so the estimated value  $|\Delta_i^{(s)}| \approx 10^{-8}$ , related to the spatial variation of photon entropy fluctuations, can be used as a constraint on the effects of inhomogeneity on primordial entropy fluctuations that might be predicted by inflationary models at earlier cosmic time. Also, the deviation from equilibrium in the initial hypersurface (proportional to  $[\Delta_i^{(r)}]^2 \approx |\Delta_i^{(s)}|$ ) might be helpful to understand the irreversibility associated with the physical processes involved in the generation of primordial perturbations [34]. These and other applications are worth to be undertaken in future research efforts.

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- [1] S. Weinberg, *Gravitation and Cosmology*, (J. Wiley, N.Y., 1972).
  - [2] S. Weinberg, *Astrophys J.* **168**, 175, (1971).
  - [3] J.V. Narlikar, *Introduction to Cosmology*, (Jones and Bartlett, 1983).
  - [4] G. Börner, *The Early Universe, Facts and Fiction*, (Springer, Berlin, 1988).
  - [5] E.W. Kolb and M.S. Turner, *The Early Universe*, (Addison-Wesley, 1990).
  - [6] T. Padmanabhan, *Formation of Structures in the Universe* (C.U.P., Cambridge, 1995).
  - [7] J. Bernstein, *Kinetic Theory in the Expanding Universe* (C.U.P., Cambridge, 1998).
  - [8] P.J.E. Peebles, *The Large Scale Structure of the Universe* (Princeton University Press, Princeton, 1980).
  - [9] H. Kodama and M. Sasaki, *Prog. Theor. Phys. Suppl.*, **78**, (1984).

- [10] G. Efstathiou, *Cosmological Perturbations*. In *Physics of the Early Universe*, Proceedings of the Thirty Sixth Scottish Universities Summer School in Physics. Eds J.A. Peacock, A.F. Heavens and A.T. Davies. IOP Publishing Ltd, Bristol, U.K., (1990).
- [11] P.T. Landsberg and D. Evans, *Mathematical Cosmology* (OUP, Oxford, 1979).
- [12] A.A. Coley and B.O.J. Tupper, *J. Math. Phys.*, **27**, 406 (1986).
- [13] M.D. Pollock and N. Caderni, *Mon. Not. R. Astr. Soc.*, **190**, 509 (1980); J.A.S. Lima and J. Tiomno, *Gen. Rel. Grav.*, **20**, 1019 (1988); R.A. Sussman, *Class. Quantum Grav.*, **9**, 1891 (1992).
- [14] V. Méndez and D. Pavón, *Mon. Not. R. Astr. Soc.* **782**, 753 (1996).
- [15] G.R.F. Ellis et al., *Gen. Rel. Grav.*, **10**, 931 (1983); G.R.F. Ellis and S.R. Matravars, *Class. Quantum Grav.*, **7**, 1869 (1990); R. Maartens and F.P. Wolvaardt, *Class. Quantum Grav.*, **11**, 203 (1994).
- [16] W.A. Hiscock and J. Salmonson, *Phys. Rev. D*, **43**, 3249 (1991).
- [17] R. Maartens and J. Triginer, *Phys. Rev. D* **56**, (1997) 4640.
- [18] W.A. Hiscock and L. Lindblom, *Phys. Rev. D* **31**, 725 (1985).
- [19] W.A. Hiscock and L. Lindblom, *Phys. Rev. D* **35**, 3723 (1987).
- [20] W. Israel, *Ann. Phys. (NY)*, **100**, 310 (1976).
- [21] W. Israel and J. Stewart, *Ann. Phys. (NY)*, **118**, 341 (1979).
- [22] D. Pavón, D. Jou and J. Casas-Vázquez, *Ann. Inst. H. Poincaré. Ser. A*, **36**, 79 (1982).
- [23] D. Jou, J. Casas-Vázquez and G. Lebon, *Extended Irreversible Thermodynamics* 2nd edition (Springer, Berlin, 1996).
- [24] W.A. Hiscock and L. Lindblom, *Contemporary Mathematics*, **71**, 181 (1991).
- [25] N. Udey and W. Israel, *Mon. Not. R. Astr. Soc.*, **199**, 1137 (1982).
- [26] D. Pavón, D. Jou and J. Casas-Vázquez, *J. Phys. A: Math. Gen.* **16**, 775 (1983).
- [27] D. Jou and D. Pavón, *Astrophys. J.*, **291**, 447 (1985).
- [28] W. Zimdahl, *Phys. Rev. D*, **53**, 5483, (1996).
- [29] M.A. Schweizer, *J. Phys. A: Math. Gen.* **17**, 2859, (1984). See also: M.A. Schweizer, *Astron. Astrophys.* **151**, 79 (1985).
- [30] D. Kramer, H. Stephani, M.A.H. MacCallum, E. Herlt, *Exact Solutions of Einstein's Field Equations*, (CUP, Cambridge, 1980).
- [31] A. Krasinski, *Inhomogeneous Cosmological Models* (CUP, Cambridge, 1997).
- [32] R.A. Sussman and J. Triginer, submitted to *Class. Quantum Grav.*, **16**, 167, (1999).
- [33] R.A. Sussman, *Class. Quantum Grav.*, **15**, 1759, (1998).
- [34] A. Linde, *Particle Physics and Inflationary Cosmology*. Series of Contemporary Concepts in Physics, Harwood Academic Publishers, Fifth printing, 1996.
- [35] S.W. Goode and J. Wainwright, *Phys. Rev. D* **26**, 3315 (1982).
- [36] C. Eckart, *Phys. Rev.* **58**, 919 (1940).
- [37] L. Landau and E.M. Lifshitz, *Mécanique des Fluides*, (MIR, Moscou, 1971).
- [38] J.V. Arnau, M.J. Fullana, L. Monreal and D. Sáez, *Astrophys. J.* **402**, 359 (1993).
- [39] Waterloo Maple Inc., Waterloo, Ontario, Canada.

FIG. 1. **The equations  $\Gamma = 0$  and  $\Psi = 0$ .** Implicit plots of the solution of  $\Gamma = 0$  (figure 1a) and  $\Psi = 0$  (figure 1b), in terms of  $\Delta_i^{(m)}$ ,  $\Delta_i^{(r)}$  and  $\log_{10}(y)$ , with  $\Gamma, \Psi$  given by (19). The grid marks the initial surface  $y = 1$ , and is not intersected by the surfaces  $\Gamma = 0, \Psi = 0$ , illustrating that  $\Gamma$  and  $\Psi$  have no zeroes in the evolution range  $y > 1$  for density contrasts bound by  $-1 \leq \Delta_i^{(m)} \leq 1$  and  $-1 \leq \Delta_i^{(r)} \leq 1$ . We have considered  $n_i^{(r)}/n_i^{(m)} \approx 10^9$ ,  $m$  to be the mass of a proton and  $T_i \approx 10^6 \text{K}$ , hence we have made the approximation (23c):  $\epsilon \approx 10^{-3} T_i \approx 10^3$ . These (and the remaining) plots were obtained with the help of the symbolic computing package Maple V[39].

FIG. 2. **The ratio of relaxation vs Hubble times for the case  $\Delta_i^{(s)} = 0$ .** This plot displays the ratio  $\tau/t_H$ , as a function of  $\log_{10}(y)$  and  $\log_{10}(|\Delta_i^{(r)}|)$  (denoted as “log(Dr)”), where  $\tau$  is the relaxation time and  $t_H = 3/\Theta$  is the Hubble expansion time, given by (22) and (29) under the condition  $\Delta_i^{(s)} = 0$ . This ratio is less than unity for all  $y$ , and  $|\Delta_i^{(r)}| < 1$  thus violating (32b) and (32c).

FIG. 3. **The equations  $\Phi = 0$  and  $\sigma = 0$ .** Implicit plots of the solution of  $\Phi = 0$  (figure 3a) and  $\sigma = 0$  (figure 3b), in terms of  $\Delta_i^{(m)}$ ,  $\Delta_i^{(r)}$  and  $\log_{10}(y)$ . The “wall” where the surfaces  $\sigma = 0$  and  $\Phi = 0$  occur for values greater than  $y = 10^2$  corresponds to the line  $\frac{3}{4}\Delta_i^{(r)} = \Delta_i^{(m)}$  (or  $\Delta_i^{(s)} = 0$ ). The plots also illustrate that the values of  $\Delta_i^{(r)}, \Delta_i^{(m)}$  where a zero of  $\sigma$  occurs are clearly distinct from those associated with a zero of  $\Phi$ , leading to conditions (37). The vertical height  $y = 10^2$  is displayed in figures 3a and 3b, illustrating that the values of  $\Delta_i^{(r)}, \Delta_i^{(m)}$  with  $10^2 < y_* < 10^3$  are very close to  $\Delta_i^{(s)} = 0$ .

FIG. 4. **Allowed values of  $\Delta_i^{(r)}$ ,  $\Delta_i^{(s)}$  for  $\sigma = 0$ .** The figure displays the plane  $\Delta_i^{(r)}$ ,  $\Delta_i^{(s)}$  associated with figures 3. The regions for which a zero of  $\sigma$  occurs for  $\Phi \neq 0$  are shown with a gray shadow, while the blank regions are “forbidden” areas corresponding to zeroes of  $\Phi$ . The values of  $\Delta_i^{(r)}$ ,  $\Delta_i^{(s)}$  in the gray areas comply with conditions (37) and so satisfy conditions (31a), (31b) and (31c). The level curve  $\log_{10}(y) = 2.4$  has been qualitatively sketched. This curve is extremely close ( $\approx 10^{-8}$ ) to the diagonal line corresponding to  $\frac{3}{4}\Delta_i^{(r)} = \Delta_i^{(s)}$ , marked by the letter “A” (the letter “B” marks the line  $\Delta_i^{(r)} = -\Delta_i^{(s)}$ ). By following the analogy with Perturbation Theory (see section 8), the line “A” can be associated with “adiabatic” initial conditions, while “isocurvature” initial conditions would be characterized by the horizontal axis  $\Delta_i^{(r)} = 0$ . However, the values of  $y = y_D$  for the latter condition are much smaller than  $y = 10^{2.4}$  for all  $|\Delta_i^{(s)}| \neq 0$  (see figure 3b). Hence, initial conditions with  $\Delta_i^{(s)} \neq 0$  and leading to the right decoupling temperature (close to the level curve  $\log_{10}(y) = 2.4$ ) cannot be termed “isocurvature”, but rather “quasi-adiabatic” initial conditions.

FIG. 5. **The decoupling temperature  $T = T_D = 4 \times 10^3$  K for  $y = y_D$ .** This figure displays the implicit plot of equation (43) in terms of  $\Delta_i^{(m)}$ ,  $\Delta_i^{(r)}$  and  $\log_{10}(y)$ . The values of  $y$  for this temperature value are clearly shown to be very close to  $y = y_D = 10^{2.4}$ .

FIG. 6. **The equation  $\sigma = 0$  for  $y = y_D$ .** Implicit plot of the solution to the equation  $\sigma = 0$  in terms of  $\log_{10}(|\Delta_i^{(s)}|)$  and  $\log_{10}(|\Delta_i^{(r)}|)$ , denoted as “log(Ds)” and “log(Dr)”. This plot illustrates the constraints on the orders of magnitude of these quantities under the condition that a zero of  $\sigma$  occurs for  $y = y_D \approx 10^{2.4}$ . For values of  $|\Delta_i^{(r)}| \approx 10^{-4}$ , compatible with observed anisotropy of the CBR, we have the following estimated value:  $|\Delta_i^{(s)}| \approx 10^{-8}$ . This can be a constraint on entropy fluctuations in primordial quasi adiabatic perturbations.

FIG. 7. **Comparison between  $\tau$ , the timescales of Thomson ( $t_\gamma$ ) and Compton ( $t_c$ ) scatterings and  $t_H$ .** The figure depicts the ratios  $\log_{10}(\tau/t_H)$ ,  $\log_{10}(t_c/t_H)$  and  $\log_{10}(t_\gamma/t_H)$  along the range  $0 < \log_{10}(y) < 3$  for  $\Delta_i^{(m)} = (3/4)\Delta_i^{(r)} - \Delta_i^{(s)}$  with  $\Delta_i^{(r)} = 10^{-4}$  and  $10^{-8} < \Delta_i^{(s)} < 10^{-4}$ . The curves that branch out correspond to  $\log_{10}(\tau/t_H)$  for the displayed values of  $\Delta_i^{(s)}$ . The thick curves are  $\log_{10}(t_c/t_H)$  (above) and  $\log_{10}(t_\gamma/t_H)$  (below) for the same values of  $\Delta_i^{(s)}$  and  $\Delta_i^{(r)}$ . Notice that  $\tau$  is much more sensitive to changes in  $\Delta_i^{(s)}$  than  $t_\gamma$  and  $t_c$ . It is clear from the figure that  $\tau \approx t_\gamma$  for  $\Delta_i^{(s)} \approx 10^{-8}$  for temperatures near the decoupling temperature, obtained from the condition  $t_H = t_\gamma$  and closely matching  $T_D \approx 4 \times 10^3$  K. (marked by  $\log_{10}(y) \approx 2.4$ ). The figure also reveals that  $\tau$  and  $t_c$  are of the same order of magnitude for higher temperatures closer to  $t = t_i$ . This is consistent with the fact that Compton scattering is the dominant radiative process in this temperature range.



















